The Fréchet Contingency Array Problem is Max-Plus Linear

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Abstract

In this paper we show that the so-called array Fréchet problem in Probability/Statistics is $(\max, +)$-linear. The upper bound of Fréchet is obtained using simple arguments from residuation theory and lattice distributivity. The lower bound is obtained as a loop invariant of a greedy algorithm. The algorithm is based on the max-plus linearity of the Fréchet problem and the Monge property of bivariate distribution.

Keywords: Max-plus algebra, Fréchet bounds.

1 Introduction

As a preliminary remark the author would like this paper to be a modest tribute to the work of Maurice Fréchet in Statistics. The work has been started at the occasion of his 130th birthday and the 100th anniversary of his stay in Nantes (the town the author is living in) as professor in Mathematics.

In this paper it is shown that the tropical or max-plus semiring $\mathbb{R}_{\max}$ (i.e. the set $\mathbb{R}$ of real numbers with max as addition and $+$ as multiplication, see the precise definition in subsection 2.2) is the underlying algebraic structure which is well suited to the Fréchet contingency (or correlation) array problem [4]. In other words the Fréchet problem is a tropical problem which thus has its place in the new trends of idempotent mathematics founded by V. P. Maslov and its collaborators in the 1980s (see e.g.[7] and references therein).

From this main result the Fréchet upper bound is derived by residuation and the distributivity property of $\mathbb{R}_{\max}$ as a lattice. The Fréchet lower bound is obtained as a loop invariant of a greedy algorithm. This algorithm is based on the tropical nature of the Fréchet problem and the Monge property of bivariate distribution.

1.1 Organization of the paper

The paper is written to be sef-contained. Thus, in Section 2 we introduce main notations used in the paper, we restate the Fréchet array problem and its bounds. We define the tropical semiring $\mathbb{R}_{\max}$ and recall basic results on residuation theory and lattices. In Section 3 we prove the main result of the paper that is the Fréchet array problem is max-plus linear in the space of cumulative distribution functions (see Theorem 3.1). From this result in Section 4 we derive the Fréchet bounds using new approaches. The upper bound is derived from residuation theory and the lattice distributivity property of the max-plus semiring $\mathbb{R}_{\max}$ (see Corollary 4.1). The lower bound is obtained as the loop invariant of a greedy algorithm (see Proposition 4.1). In Section 5 we conclude this work.

2 Preliminaries

In this Section we recall basic results concerning Fréchet array problem and the max-plus semi-ring $\mathbb{R}_{\max}$.

2.1 The Fréchet contingency array problem and its solution

This problem is described in e.g. [4]. Let n be an integer ≥ 1. The set $\mathsf{Mat}_{nm}(\mathbb{R}_+)$ denotes the set of $n \times m$ matrices which entries are nonnegative real numbers. We define the partial order $\overset{\mathcal{D}}{\preceq}$ on $\mathsf{Mat}_{nn}(\mathbb{R}_+)$ as follows:

$$A = [a_{i,j}] \overset{\mathcal{D}}{\preceq} B = [b_{i,j}] \overset{\mathsf{def}}{\Leftrightarrow} \forall i, j,\ \sum_{l=1}^{i}\sum_{k=1}^{j} a_{l,k} \leq \sum_{l=1}^{i}\sum_{k=1}^{j} b_{l,k}. \tag{1}$$

Introducing the fundamental $n \times n$ matrix:

$$D \overset{\mathsf{def}}{=} [1_{\{i \leq j\}}], \tag{2}$$

where $1_{\{i \leq j\}} = 1$ if $i \leq j$ and 0 otherwise, the partial order $\overset{\mathcal{D}}{\preceq}$ can be rewritten as follows:

$$A \overset{\mathcal{D}}{\preceq} B \Leftrightarrow DAD^T \leq DBD^T \text{ (entrywise)} \tag{3}$$

where $()^T$ denotes the transpose operator.

Let $p, q \in \mathsf{Mat}_{n1}(\mathbb{R}_+)$ such that $\sum_{i=1}^n p_i = \sigma = \sum_{j=1}^n q_j$. Without loss of generality we can assume:

$$\sigma = 1.$$

The problem of Fréchet is then to find (if exist) the maximum and the minimum element w.r.t the partial order $\overset{\mathcal{D}}{\preceq}$ of the subset of $\mathsf{Mat}_{nn}(\mathbb{R}_+)$:

$$\mathcal{H}(p,q) \overset{\mathsf{def}}{=} \{F \in \mathsf{Mat}_{nn}(\mathbb{R}_+) | F \text{ satisfies } (\mathbf{F1}) \text{ and } (\mathbf{F2})\} \tag{4}$$

with:
($\mathbf{F1}$). $F\underline{1} = p$,
($\mathbf{F2}$). $\underline{1}^T F = q^T$.
Where $\underline{1}$ denotes the n-dimensional vector which coordinates are 1's.

Fréchet proved that there exist a maximum element, $\bigvee_{\overset{\mathcal{D}}{\preceq}} \mathcal{H}(p,q) \overset{\mathsf{not.}}{=} F_{\max}$, and a minimum element, $\bigwedge_{\overset{\mathcal{D}}{\preceq}} \mathcal{H}(p,q) \overset{\mathsf{not.}}{=} F_{\min}$ such that:

$$(DF_{\max}D^T)_{i,j} = \min((Dp)_i, (q^T D^T)_j) \overset{\mathsf{not.}}{=} (\overline{F}_{\max})_{i,j}, \tag{5a}$$

and

$$(DF_{\min}D^T)_{i,j} = \max(0, (Dp)_i + (q^T D^T)_j - \sigma) \overset{\mathsf{not.}}{=} (\overline{F}_{\min})_{i,j} \tag{5b}$$

for all $i, j = 1, \ldots, n$.

2.2 The max-plus semiring $\mathbb{R}_{\max}$

Let $\mathbb{R}$ be the field of real numbers. The max-plus semiring $\mathbb{R}_{\max}$ is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the addition $\oplus : (a,b) \mapsto a \oplus b \overset{\mathsf{def}}{=} \max(a,b)$ and the multiplication $\odot : (a,b) \mapsto a \odot b \overset{\mathsf{def}}{=} a + b$. The neutral element for $\oplus$ is $\mathbb{0} := -\infty$ and the neutral element for $\odot$ is $\mathbb{1} := 0$. The max-plus semiring is said to be an idempotent semiring because the addition is idempotent, i.e. $a \oplus a = a$, $\forall a$.

The idempotent semiring $\mathbb{R}_{\max}$ or idempotent semirings isomorphic to $\mathbb{R}_{\max}$ has many applications in discrete mathematics, algebraic geometry, computer science, computer languages, linguistic problems, optimization theory, discrete event systems, fuzzy theory (see e.g. [1], [3], [6], [8]).

2.3 Order properties of $\mathbb{R}_{\max}$ and residuation

Let us consider the max-plus semiring $\mathbb{R}_{\max}$ already defined in the introduction. The binary relation $\mathcal{R}$ defined by: $a\mathcal{R}b \overset{\text{def}}{\Leftrightarrow} a \oplus b = b$ coincides with the standard order $\leq$ on $\mathbb{R}$. We denote $\overline{\mathbb{R}}_{\max}$ the semiring completed by adjoining to $\mathbb{R}_{\max}$ a $\top := +\infty$ element which satisfies $a \oplus \top = \top$, $\forall a$, $a \odot \top = \top$, $\forall a \neq \mathbb{0}$, and $\mathbb{0} \odot \top = \mathbb{0}$. This is mainly motivated by the fact that some of the further results can be stated in a simpler way in $\overline{\mathbb{R}}_{\max}$.

The completed max-plus semiring $\overline{\mathbb{R}}_{\max}$ is a complete sup-semilattice, i.e. $\forall A \subseteq \overline{\mathbb{R}}\ \oplus A \overset{\text{def}}{=} \underset{x \in A}{\oplus} x$ exists in $\overline{\mathbb{R}} \overset{\text{def}}{=} \mathbb{R} \cup \{-\infty, +\infty\}$. This implies that $\overline{\mathbb{R}}_{\max}$ is also a complete inf-semilattice because $\wedge A = \oplus\{x \in \overline{\mathbb{R}} | \forall a \in A,\ x \leq a\}$. Thus, $\overline{\mathbb{R}}_{\max}$ is a complete lattice. Finally, let us mention that $\overline{\mathbb{R}}_{\max}$ is a distributive lattice, i.e.:

$$a \oplus (b \wedge c) = (a \wedge b) \oplus (a \wedge c). \tag{6}$$

As we will see in the sequel this property will be of particular importance. Let us define left and right division in $\overline{\mathbb{R}}_{\max}$ by: $b/a \overset{\text{def}}{=} \oplus\{x \in \overline{\mathbb{R}} | x \odot a \leq b\}$ and $a \backslash b \overset{\text{def}}{=} \oplus\{x \in \overline{\mathbb{R}} | a \odot x \leq b\}$. Left and right division coincide with the usual subtraction to which we add the following properties: $\mathbb{0} \backslash a = \top$, $\top \backslash a = \mathbb{0}$ if $a \neq \top$, $\top$ otherwise (similar formulae for $/$).

We extend operations and binary relations from scalars to matrices as follows. If $A = [a_{i,j}]$, $B = [b_{i,j}]$ then: $A \leq B$ (entrywise) $\Leftrightarrow \forall i, j,\ a_{i,j} \leq b_{i,j}$, $A \oplus B = [a_{i,j} \oplus b_{i,j}]$, $A \wedge B = [a_{i,j} \wedge b_{i,j}]$, and $A \odot B$ denotes the matrix such that its entry (i,j), $(A \odot B)_{i,j}$, is: $(A \odot B)_{i,j} = \oplus_k a_{i,k} \odot b_{k,j}$. We obviously have: $(A \odot B)^T = B^T \odot A^T$. We also extend the divisions to (possibly rectangular) matrices with suitable dimension:

$$(A \backslash B)_{i,j} \overset{\text{def}}{=} (\oplus\{X | A \odot X \leq B\})_{i,j} = \wedge_k a_{k,i}\ b_{k,j}, \tag{7a}$$

$$(D/C)_{i,j} \overset{\text{def}}{=} (\oplus\{X | X \odot C \leq D\})_{i,j} = \wedge_l d_{i,l}/c_{j,l}. \tag{7b}$$

The $\oplus$ in the formulae (7a) and (7b) corresponds to the supremum w.r.t entrywise order between matrices. The application $Y \mapsto A \backslash Y$ (resp. $Y \mapsto Y/C$) is called the *residuated mapping* of the application $X \mapsto A \odot X$ (resp. $X \mapsto X \odot C$).

For more details on max-plus algebra and residuation theory we refer the reader to e.g [1, Chp. 4 and references therein].

3 Main Result

We begin this Section by the following fundamental lemma.

Lemma 3.1 *Let* $(u^j)_{j=1}^m$ *be* m *elements of* $\mathsf{Mat}_{n1}(\mathbb{R}_+)$. *Define* $s^j = \sum_{k=1}^j u^k$. *Then,*

$$[u^1 \cdots u^n]\,\underline{1} = [s^1 \cdots s^n] \odot \underline{\mathbb{1}} \tag{8}$$

where $\underline{\mathbb{1}}$ *denotes the* n*-dimensional vector which all components are* $\mathbb{1}$*'s.*

Proof. Let us remark that: $[u^1 \cdots u^n]\,\underline{1} = s^n$. Because the vectors u^j have all their coordinates nonnegative: $s^1 \leq \cdots \leq s^n$ (componentwise), which is equivalent to: $s^n = s^1 \oplus \cdots \oplus s^n$. Now, we just have to remark that: $s^1 \oplus \cdots \oplus s^n = [s^1 \cdots s^n] \odot \underline{\mathbb{1}}$ which ends the proof of the result. □

Remark 3.1 *In the previous Lemma define* $U = [u^1 \cdots u^n]$ *and* $S = [s^1 \cdots s^n]$. *We remark that* $S = UD^T$, *recalling that* D *is the matrix defined by (2). Thus, relation (8) can be rewritten:*

$$U\underline{1} = (UD^T) \odot \underline{\mathbb{1}}. \tag{9}$$

Theorem 3.1 (Main Result) *Let us consider a matrix* $F \in \mathsf{Mat}_{nn}(\mathbb{R}_+)$. *Let* p *and* q *be two elements of* $\mathsf{Mat}_{n1}(\mathbb{R}_+)$. *Then,*

$$F \in \mathcal{H}(p,q) \Longleftrightarrow \begin{cases} (DFD^T) \odot \underline{\mathbb{1}} & = Dp \\ \underline{\mathbb{1}}^T \odot (DFD^T) & = q^T D^T \end{cases}$$

Proof. $F \in \mathcal{H}(p,q) \Leftrightarrow \begin{cases} F\underline{1} & = p \\ \underline{1}^T F & = q^T \end{cases}$.

Because matrix D is invertible so is D^T and:

$$\begin{cases} F\underline{1} & = p \\ \underline{1}^T F & = q^T \end{cases} \Longleftrightarrow \begin{cases} DF\underline{1} & = Dp & \text{(eq 1)} \\ \underline{1}^T F D^T & = q^T D^T & \text{(eq 2)}. \end{cases}$$

For (eq 1). We apply Lemma 3.1 with $u^j := (DF)_{.,j}$ and $s^j = (DFD^T)_{.,j}$ be the jth column vectors of matrices DF and DFD^T, respectively. We obtain the following equality: $Dp = DF\underline{1} = (DFD^T) \odot \underline{\mathbb{1}}$.
For (eq 2). We apply Lemma 3.1 with $u^j := (DF^T)_{.,j}$ and $s^j := (DF^T D^T)_{.,j}$. We have: $DF^T\underline{1} = (q^T D^T)^T = (DF^T D^T) \odot \underline{\mathbb{1}}$. By definition of $()^T$ we have: $q^T D^T = \underline{\mathbb{1}}^T \odot (DFD^T)$ which ends the proof of the Theorem. □

This result can be reformulated as follows. Let us define the following sets:

$$\mathsf{Dist}_{n1} \stackrel{\mathsf{def}}{=} D\ \mathsf{Mat}_{n1}(\mathbb{R}_+) = \{Dx,\ x \in \mathsf{Mat}_{n1}(\mathbb{R}_+)\} \tag{10}$$

and

$$\mathsf{Dist}_{nn} \stackrel{\mathsf{def}}{=} D\ \mathsf{Mat}_{nn}(\mathbb{R}_+)\ D^T = \{DXD^T,\ X \in \mathsf{Mat}_{nn}(\mathbb{R}_+)\} \tag{11}$$

and for all $\overline{P}, \overline{Q} \in \mathsf{Dist}_{n1}$:

$$\mathbb{H}(\overline{P}, \overline{Q}) \stackrel{\mathsf{def}}{=} \{\overline{F} \in \mathsf{Dist}_{nn} | \overline{F} \odot \underline{\mathbb{1}} = \overline{P} \text{ and } \underline{\mathbb{1}}^T \odot \overline{F} = \overline{Q}^T\}. \tag{12}$$

Then, Theorem 3.1 states that $\forall p, q \in \mathsf{Mat}_{n1}(\mathbb{R}_+)$ and $\forall F \in \mathsf{Mat}_{nn}(\mathbb{R}_+)$:

$$F \in \mathcal{H}(p, q) \Leftrightarrow DFD^T \in \mathbb{H}(Dp, Dq). \tag{13}$$

Or, equivalently:

$$D\ \mathcal{H}(p, q)\ D^T = \mathbb{H}(Dp, Dq). \tag{14}$$

4 New approach for the Fréchet bounds

From our main result, Theorem 3.1, we obtain Fréchet bounds by methods which seem to be new to the best knowledge of the author.

4.1 Upper bound

The Fréchet upper bound is obtained as a direct consequence of Theorem 3.1.

Corollary 4.1 (Fréchet upper bound) *Let p and q be two elements of $\mathsf{Mat}_{n1}(\mathbb{R}_+)$. Under the condition that $p^T\underline{1} = q^T\underline{1}$ the set $\mathcal{H}(p, q)$ is not empty and the upper Fréchet bound $F_{\max}$ is such that $DF_{\max}D^T \stackrel{\mathsf{not.}}{=} \overline{F}_{\max}$ is the greatest sub-solution of the following max-plus linear system of equations:*

$$\begin{cases} \overline{F} \odot \underline{\mathbb{1}} & = Dp \\ \underline{\mathbb{1}}^T \odot \overline{F} & = q^T D^T \end{cases} \tag{15}$$

that is:

$$\overline{F}_{\max} = ((Dp)/\underline{\mathbb{1}}) \wedge (\underline{\mathbb{1}}^T \backslash (q^T D^T)).$$

Proof. Let us study (15) when replacing $=$ by $\leq$. Then, by definition of $/$ we have: $\overline{F} \odot \underline{\mathbb{1}} \leq Dp \Leftrightarrow \overline{F} \leq (Dp)/\underline{\mathbb{1}}$. And by definition of $\backslash$ we have: $\overline{F} \leq \underline{\mathbb{1}}^T \backslash (q^T D^T)$. The two previous inequalities are equivalent to: $\overline{F} \leq ((Dp)/\underline{\mathbb{1}}) \wedge (\underline{\mathbb{1}}^T \backslash (q^T D^T)) = \overline{F}_{\max}$. Now, we have to prove that (A): $\overline{F}_{\max} \odot \underline{\mathbb{1}} = Dp$ and (B): $\underline{\mathbb{1}}^T \odot \overline{F}_{\max} = q^T D^T$.

Let us prove (A).
For all $i = 1, \dots, n$ we write:

$$\begin{aligned}
(\overline{F}_{\max} \odot \underline{\mathbb{1}})_i &= \oplus_{j=1}^{n}((Dp)/\underline{\mathbb{1}}) \wedge (\underline{\mathbb{1}}^T \backslash (q^T D^T))_{i,j} \odot \mathbb{1} \\
&= \oplus_{j=1}^{n}(Dp)_i \wedge (q^T D^T)_j \\
&= (Dp)_i \wedge (\oplus_{j=1}^{n}(q^T D^T)_j) \left(\text{by lattice distributivity (6)}\right) \\
&= (Dp)_i \wedge (q^T D^T)_n \left(\forall j\colon\ (q^T D^T)_j \leq (q^T D^T)_n\right) \\
&= (Dp)_i \wedge (Dp)_n \left((b^T D^T)_n = q^T \underline{1} = p^T \underline{1} = (Dp)_n\right) \\
&= (Dp)_i \left(\forall i\colon\ (Dp)_i \leq (Dp)_n\right).
\end{aligned}$$

We prove (B) similarly. Hence the result is now achieved. □

4.2 Lower bound

In this section we obtain Fréchet lower bound by a greedy algorithm based on max-plus linearity of the Fréchet problem and the well-known Monge property (see e.g. [2]) of elements of the set Dist_{nn} defined by (11), that is for all $\overline{F} \in \mathsf{Dist}_{nn}$:
(M). $\forall i, j = 0, \dots n-1$: $\overline{F}_{i,j} \odot \overline{F}_{i+1,j+1} \geq \overline{F}_{i,j+1} \odot \overline{F}_{i+1,j}$,
with the convention that $\forall k\ \overline{F}_{0,k} = \overline{F}_{0,k} = 0$.
Let $p, q \in \mathsf{Mat}_{n1}(\mathbb{R}_+)$ be two vectors such that $p^T\ \underline{1} = q^T\ \underline{1} = 1$. Let us consider the following algorithm.
`Lower`(n,p,q)
$\forall i = 1, \dots, n$, $\overline{F}_{i,n} := (Dp)_i$; (a)
$\forall j = 1, \dots, n$, $\overline{F}_{n,j} := (q^T D^T)_j$; (b)
For $j = n-1$ to 1 do

For $i = n-1$ to 1 do

$$\overline{F}_{i,j} := \overline{F}_{i+1,j+1}^{-1} \odot (\overline{F}_{i,j+1} \odot \overline{F}_{i+1,j}) \oplus \mathbb{1}$$

end

end.

Proposition 4.1 *The algorithm* `Lower` *computes the Lower bound of the Fréchet contingency array problem.*

Proof. The initial conditions (a) and (b) of the algorithm `Lower` come from the max-plus linearity of the Fréchet problem and that the Monge property (**M**) implies:

$$\forall i \leq i', j \leq j',\ \overline{F}_{i,j} \leq \overline{F}_{i',j'}.$$

The proof is obtained by recurrence (see e.g. the detailed proof of this result by Fréchet himself [5, p. 13])

Denoting $\alpha_l = (Dp)_l$, $\beta_k = (q^T D^T)_k$ we have to prove that the loop invariant of the algorithm `Lower` corresponds to the Fréchet lower bound, i.e. $\forall l, k$: $\overline{F}_{l,k} = 1^{-1} \odot \alpha_l \odot \beta_k \oplus \mathbb{1}$.

It is easy to see that the previous relation is true for $l = n$ with $k = 1, \dots, n$ and for $l = 1, \dots, n$ with $k = n$. Now, let us assume that the loop invariant is true for $(k,l) \geq (i,j)$, $(k,l) \neq (i,j)$. We have:

$$\begin{aligned}\overline{F}_{i,j+1} \odot \overline{F}_{i+1,j} &= (1^{-1} \odot \alpha_i \odot \beta_{j+1} \oplus \mathbb{1}) \odot (1^{-1} \odot \alpha_{i+1} \odot \beta_j \oplus \mathbb{1}) \\ &= 2^{-1} \odot \alpha_i \odot \alpha_{i+1} \odot \beta_j \odot \beta_{j+1} \\ &\oplus 1^{-1} \alpha_i \odot \beta_{j+1} \oplus 1^{-1} \odot \alpha_{i+1} \odot \beta_j \oplus \mathbb{1}.\end{aligned}$$

Because $(\mathbb{R}_+, \leq)$ is a totally ordered lattice:

$$\overline{F}_{i+1,j+1} = 1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1} \oplus \mathbb{1} \ \in \{1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1}, \mathbb{1}\}$$

Thus, we have two cases to study.

<u>1rst case</u>: $\overline{F}_{i+1,j+1} = 1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1}$
Let us compute:

$$\begin{aligned}\overline{F}^{-1}_{i+1,j+1} \odot \overline{F}_{i,j+1} \odot \overline{F}_{i+1,j} &= 1^{-1} \odot \alpha_i \odot \beta_j \oplus \alpha_i \odot \alpha^{-1}_{i+1} \oplus \beta_j \odot \beta^{-1}_{j+1} \\ &\oplus 1 \odot (\alpha_{i+1} \odot \beta_{j+1})^{-1}.\end{aligned}$$

Then, we just have to remark that: $\alpha_i \odot \alpha^{-1}_{i+1} = p^{-1}_{i+1} \leq \mathbb{1}$, $\beta_j \odot \beta^{-1}_{j+1} = q^{-1}_{j+1} \leq \mathbb{1}$ and $\overline{F}_{i+1,j+1} = 1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1} \Leftrightarrow 1 \odot (\alpha_{i+1} \odot \beta_{j+1})^{-1} \leq \mathbb{1}$. Thus,

$$\overline{F}_{i,j} = \overline{F}^{-1}_{i+1,j+1} \odot \overline{F}_{i,j+1} \odot \overline{F}_{i+1,j} \oplus \mathbb{1} = 1^{-1} \odot \alpha_i \odot \beta_j \oplus \mathbb{1}.$$

<u>2nd case</u>: $\overline{F}_{i+1,j+1} = \mathbb{1}$

$$\begin{aligned}\overline{F}^{-1}_{i+1,j+1} \odot \overline{F}_{i,j+1} \odot \overline{F}_{i+1,j} &= \overline{F}_{i,j+1} \odot \overline{F}_{i+1,j} \\ &= 2^{-1} \odot \alpha_i \odot \alpha_{i+1} \odot \beta_j \odot \beta_{j+1} \oplus 1^{-1} \alpha_i \odot \beta_{j+1} \\ &\oplus 1^{-1} \odot \alpha_{i+1} \odot \beta_j \oplus \mathbb{1}\end{aligned}$$

which could be rewritten as follows:

$$\overline{F}_{i,j} = 1^{-1} \odot \alpha_i \odot \beta_j \odot (1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1} \oplus q_{j+1} \oplus p_{i+1}) \oplus \mathbb{1}.$$

We remark that $q_{j+1}, p_{i+1} \geq \mathbb{1}$. We also note that $\overline{F}_{i+1,j+1} = \mathbb{1} \Rightarrow 1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1} \leq \mathbb{1}$. Thus, we deduce because $\odot$ is non-decreasing and the definition of the standard order $\leq$ that:

$$\overline{F}_{i,j} \geq 1^{-1} \odot \alpha_i \odot \beta_j \oplus \mathbb{1}.$$

On the other hand $\overline{F}_{i+1,j+1} = \mathbb{1} \Rightarrow 1^{-1} \odot \alpha_i \odot \beta_j \leq p_{i+1}^{-1} \odot q_{j+1}^{-1}$. And we deduce that:

$$\begin{aligned} \overline{F}_{i,j} &\leq p_{i+1}^{-1} \odot q_{j+1}^{-1}(1^{-1} \odot \alpha_{i+1} \odot \beta_{j+1} \oplus q_{j+1} \oplus p_{i+1}) \oplus \mathbb{1} \\ &= 1^{-1} \odot \alpha_i \odot \beta_j \oplus p_{i+1}^{-1} \oplus q_{j+1}^{-1} \oplus \mathbb{1} \\ &= 1^{-1} \odot \alpha_i \odot \beta_j \oplus \mathbb{1} \text{ (because } p_{i+1}^{-1}, q_{j+1}^{-1} \leq \mathbb{1}) \end{aligned}$$

We conclude because $\leq$ is antisymmetric.

□

5 Conclusion

In this paper we proved that the Fréchet correlation array problem is max-plus linear in the space of cumulative distribution function Dist_{nn} defined by (11). This remark leads to new methods to obtain the Fréchet bounds.

As a further work we would like to extend results of the paper to the continuous case based on the remark that :

$$\int_{\mathbb{R}} f(x,y)dy = \sup_{z \in \mathbb{R}} \left(\int_{-\infty}^{z} f(x,y)dy \right)$$

for all nonnegative functions f such that $\int_{\mathbb{R}} f(x,y)dy$ exists.

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